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TORSIONAL SURFACE WAVE IN SELF-REINFORCED LAYER SANDWICHED BETWEEN TWO VISCO-ELASIC HALF-SPACES UNDER THE INITIAL STRESS

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ABSTRACT
Torsional surface waves are horizontally polarized waves possessing a remarkable characteristic that they give a twist to the medium when propagated. This paper aims to investigate the effect of initial stress on the torsional surface wave propagation in the self-reinforced layer sandwiched between two visco-elastic half spaces. Dispersion relation is obtained in closed form. Numerical results and particular cases have also been discussed with and without initial stress. Graphs have been plotted between phase velocity and wave number and it is observed that the phase velocity decreases when the angular frequency of surface waves increases and effect of initial stress in negligible.

KEYWORDS: Torsional Surface wave, Self-reinforcement, Visco-elasticity, Dispersion, Initial stress.

INTRODUCTION

For the interpretation of geophysical data it is important to understand the mechanism by which waves are propagated in layered media. The earth is considered to be a layered elastic medium such that some of its parts exhibit viscoelastic nature. In past few years attention has been given to the problems of generation and propagation of elastic waves in anisotropic elastic solids or layers of different configurations. The studies of wave propagation in an elastic media have received more attention in recent past because the need of a complete understanding of different medium characteristics with respect to mechanical shocks and vibrations. The basic literature on the propagation of elastic waves is the monograph by Ewing et al. (1957). It is well known fact that the propagation of seismic waves in anisotropic media such as composite media or viscoelastic media is basically different from the propagation in isotropic media. Lord Rayleigh (1945), in his remarkable paper, has shown that the isotropic homogenous elastic half-space does not allow a torsional surface wave to propagate. On the torsional wave propagation in a two-layered circular cylinder with imperfect bonding has been discussed by Bhattacharya (1975).

The propagation of torsional waves in a finite piezoelectric cylindrical shell has been discussed by Paul et al. (1977). Propagation of torsional waves in initially stressed cylinder has been discussed by Dey et al. (1992). Some notable works in propagation of surface waves may be cited as Akbarov et al. (2011) and Chattopadhyay et al. (2011). Recently, Dhua et al. (2013) studied the propagation of torsional wave in a composite layer overlying an anisotropic heterogeneous half-space with initial stress. The present paper studies the possibility of torsional surface waves in fibre-reinforced layer bonded between pre-stressed viscoelastic half-spaces.

Formulation and Solution of the problem

In this paper we have considered a self-rein-forced composite layer of finite thickness H sandwiched between semi-infinite viscoelastic mediums. Let us consider the cylindrical coordinate system in such a way that r-axis is in the direction of the torsional wave propagation along the common interface of the middle layer and the lower semi-infinite medium and z-axis is pointing vertically downwards and upwards as shown in fig 1.

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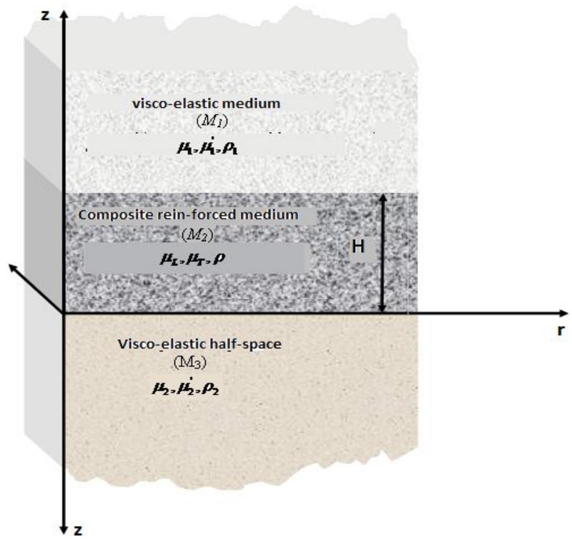

Fig.1 Geometry of the problem

Solution for upper half-space (M_1)

Let us consider u_1 , v_1 and w_1 as the displacement components in radial, azimuthally and axial directions respectively for viscoelastic upper layer. So for propagation of torsional surface wave in *r* □direction causes displacement in □□□direction only, we get

$u_1 = 0,\ w_1 = 0,$ and $v_1 = v_1(r, z, t).$

The only non-vanishing equation of motion for viscoelastic upper half-space is given by (Biot 1965)

$$\left(\mu_1 + \mu_1' \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 v_1}{\partial r^2} + \frac{\partial^2 v_1}{\partial z^2} + \frac{1}{r}\frac{\partial v_1}{\partial r} - \frac{v_1}{r^2}\right) - \frac{\partial}{\partial z}\left[\frac{P}{2}\left(\frac{\partial v_1}{\partial z}\right)\right] = \rho_1 \frac{\partial^2 v_1}{\partial t^2} \qquad (1)$$

where μ_1 □is modulus of rigidity, μ_1' □is the internal friction, ρ_1 □is the density and *P* is the initial stress acting along *r* □direction in the medium.

let us assume the solution of Eq. (1), in following form

$$v_1 = V_1(z) J_1(kr) e^{i\omega t} \qquad (2)$$

using Eq. (2) in Eq. (1), gives

$$\frac{d^2 V_1}{dz^2} - k^2 \alpha^2 V_1(z) = 0 \qquad (3)$$

where

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$$\alpha^2 = \left[\frac{1 - \dfrac{c^2}{\beta_1^2 \left(1 + \dfrac{i\omega\mu_1'}{\mu_1}\right)}}{1 - \dfrac{P}{2\mu_1 \left(1 + \dfrac{i\omega\mu_1'}{\mu_1}\right)}} \right] \quad and \quad \beta_1 = \sqrt{\frac{\mu_1}{\rho_1}}$$

the solution of Eq. (3) is

$$V_1(z) = Ee^{k\alpha z} + Fe^{-k\alpha z} \qquad (4)$$

where E and F are constants.

Taking $\lim_{z \to -\infty} V_1(z) = 0$, we finally get the solution of Eq. (1) is

$$v_1 = EJ_1(kr)e^{i\omega t + k\alpha z} \qquad (5)$$

Formulation and solution for reinforced layer (M_2)

The constitutive equation used in a self-reinforced (composite) linearly elastic model is (Belfield et al., 1983).

$$\tau_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu_T e_{ij} + \alpha^*(a_k a_m e_{km}\delta_{ij} + e_{kk}a_i a_j) + 2(\mu_L - \mu_T)(a_i a_k e_{kj} + a_j a_k e_{ki}) + \beta^* a_k a_m e_{km} a_i a_j$$

$$(i, j, k, m) = 1, 2, 3. \qquad (6)$$

Where τ_{ij} are components of stress, e_{ij} are components of infinitesimal strain, δ_{ij} Kronecker delta, a_i components of $\vec{a}$. $\vec{a} = (a_1, a_2, a_3)$ is the preferred direction of reinforcement such that $a_1^2 + a_2^2 + a_3^2 = 1$. The vector $\vec{a}$ may be a function of position. Indices take the values 1, 2, 3 and summation convention is employed. The coefficients $\lambda, \mu_T, \alpha^*, \beta^*$ and $2(\mu_L - \mu_L)$ are elastic constants with dimension of stress. μ_T can be identified as the shear modulus in transverse shear across the preferred direction and μ_L as the shear modulus in longitudinal shear in the preferred direction. α^* and β^* are specific stress components to take into account different layers for the concrete part of the composite material. In Eq. (6), the unit vector $\vec{a} = (a_1, a_2, a_3)$ gives the orientation of the family of fibres in axial z, azimuthally θ, and radial r directions respectively. On fixing the component $a_2 = 0$, we get the orientation of our choice. For some constant value of θ, the fibres initially lie in the surface and are inclined at an angle ϕ to the r axis. Therefore, the components of vector $\vec{a}$ in the cylindrical polar coordinate system are $\vec{a} = (\sin\phi, 0, \cos\phi)$. Hence, the values of components of $\vec{a}$ i.e. a_1, a_2, a_3 are $\sin\phi, 0, \cos\phi$ respectively. If u_2, v_2 and w_2 are the displacement components in radial, azimuthally and axial directions respectively, then for torsional surface waves, propagating in r –direction and causes displacement in θ –direction only, we shall assume the displacement components as

$$u_2 = u_r = 0, w_2 = u_z = 0 \; and \; v_2 = u_\theta = v_2(r, z, t). \qquad (7)$$

We have the strain displacement relation in view of Eq. (7) as

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$$e_{rr}=0, e_{\theta\theta}=0, e_{zz}=0, e_{zr}=0, e_{\theta z}=\frac{1}{2}\frac{\partial v_2}{\partial z},\ e_{r\theta}=\frac{1}{2}\left(\frac{\partial v_2}{\partial r}-\frac{v_2}{r}\right) \tag{8}$$

on fixing $a_2 = 0$ and using Eqs. (7) and (8) in Eq. (6), we have

$$\tau_{\theta z}=R\frac{\partial v_2}{\partial z}+Q\left(\frac{\partial v_2}{\partial r}-\frac{v_2}{r}\right) \text{ and } \tau_{r\theta}=S\left(\frac{\partial v_2}{\partial r}-\frac{v_2}{r}\right)+Q\frac{\partial v_2}{\partial z} \tag{9}$$

where

$$Q=a_1a_3(\mu_L-\mu_T),$$
$$R=\mu_T+a_3^2(\mu_L-\mu_T), \tag{10}$$
$$S=\mu_T+a_1^2(\mu_L-\mu_T),$$

now, the only non-vanishing equation of motion for the propagation of torsional surface wave without body force is given by

$$R\frac{\partial^2 v_2}{\partial z^2}+2Q\frac{\partial^2 v_2}{\partial r\partial z}+S\frac{\partial^2 v_2}{\partial r^2}+\frac{Q}{r}\frac{\partial v_2}{\partial z}+\frac{S}{r}\left(\frac{\partial v_2}{\partial z}-\frac{v_2}{r}\right)=\rho\frac{\partial^2 v_2}{\partial t^2} \tag{11}$$

where ρ is the density of the medium.

now, let the solution of Eq. (11) is

$$v_2=V_2(z)J_1(kr)e^{i\omega t} \tag{12}$$

where k is the wave number, ω (= $\kappa\chi$) is the circular frequency, c is the torsional wave velocity and J_1 is the Bessel's function of the first order and of the first kind.

With the help of equation (12), equation (11) reduces to

$$\frac{d^2V_2}{dz^2}+C\frac{dV_2}{dz}+DV_2=0 \tag{13}$$

where,

$$C=\frac{Q}{Rr}+\frac{2QkJ_1'(kr)}{RJ_1(kr)}, D=\frac{\rho\omega^2}{R}+\frac{SkJ_1'(kr)}{RrJ_1(kr)}+\frac{Sk^2J_1''(kr)}{RJ_1(kr)}-\frac{S}{Rr^2} \text{ and } \beta_2=\sqrt{\frac{\mu_T}{\rho}} \tag{14}$$

By using Eq. (13), we finally get the solution of the composite reinforced layer is given by

$$v_2=e^{\frac{-Cz}{2}}\left(T_1\sin\sqrt{Lz}+T_2\cos\sqrt{Lz}\right)J_1(kr)e^{i\omega t}. \tag{15}$$

where $L=D-\frac{C^2}{4}$ and C, D, T_1, T_2 are constants.

Solution for lower half-space (M_3)

Let us assume u_3 , v_3 and w_3 as the displacement components in radial, azimuthally and axial directions respectively for initially stressed viscoelastic half-space. So for torsional surface wave propagating in r –direction and causing displacement in θ–direction only, we get

$u_3=0,\ w_3=0,$ and $v_3=v_3(r,z,t).$

The only non-vanishing equation of motion for viscoelastic half-space under initial stress is given by (Biot 1965)

$$\left(\mu_2+\mu_2'\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 v_3}{\partial r^2}+\frac{\partial^2 v_3}{\partial z^2}+\frac{1}{r}\frac{\partial v_3}{\partial r}-\frac{v_3}{r^2}\right)-\frac{\partial}{\partial z}\left[\frac{P}{2}\left(\frac{\partial v_3}{\partial z}\right)\right]=\rho_2\frac{\partial^2 v_3}{\partial t^2} \tag{16}$$

where μ_2 is modulus of rigidity, μ_2' is the internal friction, ρ_2 is the density and P is the initial stress acting along r direction in the medium.

for the wave propagating along r direction, let we assume the solution of Eq. (16) as

$$v_3=V_3(z)J_1(kr)e^{i\omega t} \tag{17}$$

by using equation (17) in equation (16), we get

$$\frac{d^2V_3}{dz^2}-k^2\beta^2V_3(z)=0 \tag{18}$$

where

$$\beta^2=\left[\frac{1-\dfrac{c^2}{\beta_3^2\left(1+\dfrac{i\omega\mu_2'}{\mu_2}\right)}}{1-\dfrac{P}{2\mu_2\left(1+\dfrac{i\omega\mu_2'}{\mu_2}\right)}}\right],\qquad \beta_3=\sqrt{\frac{\mu_2}{\rho_2}} \tag{19}$$

the solution of Eq. (18) is

$$V_3(z)=Ae^{k\beta z}+Be^{-k\beta z}$$

where A and B are constants.

Taking $\lim_{z\to\infty}V_3(z)=0$, we finally get the solution of equation (16) is

$$v_3=BJ_1(kr)e^{i\omega t-k\beta z}. \tag{20}$$

Boundary conditions

The boundary conditions are as follows

(1) At $z=-H$ displacements are continuous i.e. $v_1=v_2$ at $z=-H$ (21)

(2) At $z=-H$ stresses are continuous i.e. $\tau_{\theta z}=\left(\mu_1+\mu_1'\frac{\partial}{\partial t}\right)\frac{\partial v_1}{\partial z}$ at $z=-H$ (22)

(3) At $z=0$ displacements are continuous i.e. $v_2=v_3$ at $z=0$ (23)

(4) At $z=0$ stresses are continuous i.e. $\tau_{\theta z}=\left(\mu_2+\mu_2'\frac{\partial}{\partial t}\right)\frac{\partial v_3}{\partial z}$ at $z=0$ (24)

where

$$\tau_{\theta z}=R\frac{\partial v_2}{\partial z}+Q\left(\frac{\partial v_2}{\partial r}-\frac{v_2}{r}\right)$$

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Using these boundary conditions and eliminating the arbitrary constants, we get

$$\tan\left(kH\sqrt{L_1}\right)=\frac{Rk\sqrt{L_1}\left[\left(\mu_2+i\omega\mu_2^{'}\right)k\beta-\left(\mu_1+i\omega\mu_1^{'}\right)k\alpha\right]}{Y^2-\left(\mu_1+i\omega\mu_1^{'}\right)k\alpha Y-\left(\mu_2+i\omega\mu_2^{'}\right)k\beta Y+\left(\mu_1+i\omega\mu_1^{'}\right)\left(\mu_2+i\omega\mu_2^{'}\right)k^2\alpha\beta+R^2k^2L_1} \quad (25)$$

where

$$Y=\left[\frac{RC}{2}-Q\left(k\frac{J_1^{'}(kr)}{J_1(kr)}-\frac{1}{r}\right)\right]$$

$$L_1=\left[\left(\frac{SJ_1^{''}(kr)}{RJ_1(kr)}+\frac{SJ_1^{'}(kr)}{RkrJ_1(kr)}+\frac{\rho c^2}{R}-\frac{S}{Rk^2r^2}\right)-\frac{1}{4}\left(\frac{2QJ_1^{'}(kr)}{RJ_1(kr)}+\frac{Q}{Rkr}\right)^2\right]$$

now comparing real and imaginary parts of equation (25), we get
real part as

$$\tan\left(kH\sqrt{L_1}\right)=\frac{Rk^2\sqrt{L_1}\left(\mu_2\beta-\mu_1\alpha\right)}{Y^2-kY\left(\mu_1\alpha+\mu_2\beta\right)+\left(\mu_1\mu_2-\omega^2\mu_1^{'}\mu_2^{'}\right)k^2\alpha\beta+R^2k^2L_1} \quad (26)$$

and imaginary part as

$$\tan\left(kH\sqrt{L_1}\right)=\frac{Rk^2\sqrt{L_1}\left(\mu_2^{'}\beta-\mu_1^{'}\alpha\right)}{\left(\mu_1^{'}\mu_2+\mu_1\mu_2^{'}\right)k^2\alpha\beta-kY\left(\mu_1^{'}\alpha-\mu_2^{'}\beta\right)}. \quad (27)$$

Equation (26) is the dispersion equation for propagation of torsional surface wave in a composite layer of finite thickness H sandwiched between viscoelastic half-spaces, whereas equation (27) is the damping equation associated with torsional surface wave.

Particular cases:
Case I:

When upper half-space M_1 and lower visco-elastic half space M_3 are isotropic, i.e. $\mu_1^{'}=0$ and $\mu_2^{'}=0$ with initial stress, equation (26) reduces to

$$\tan\left(kH\sqrt{L_1}\right)=\frac{Rk^2\sqrt{L_1}\left(\mu_2\beta_{11}-\mu_1\alpha_{11}\right)}{Y^2-kY\left(\mu_1\alpha_{11}+\mu_2\beta_{11}\right)+\mu_1\mu_2k^2\alpha_{11}\beta_{11}+R^2k^2L_1}. \quad (28)$$

where

$$\alpha_{11}^2=\left[\frac{1-\frac{c^2}{\beta_1^2}}{1-\frac{P}{2\mu_1}}\right] \qquad \text{and} \qquad \beta_1=\sqrt{\frac{\mu_1}{\rho_1}}$$

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$$\beta_{11}^2 = \left[\frac{1-\frac{c^2}{\beta_3^2}}{1-\frac{P}{2\mu_2}} \right] \quad \text{and} \quad \beta_3 = \sqrt{\frac{\mu_2}{\rho_2}}$$

Equation (28) is the dispersion equation for the propagation of torsional surface wave in self- reinforced layer bonded between initially stressed isotropic half-spaces.

Case II:

When composite layer M_2 is isotropic, i.e. $\mu_L = \mu_T = \mu_0$, the dispersion equation (26) becomes

$$\tan\left(kH\sqrt{L_2}\right) = \frac{\mu_0 k^2 \sqrt{L_2}\left(\mu_2\beta - \mu_1\alpha\right)}{\left(\mu_1\mu_2 - \omega^2\mu_1^{'}\mu_2^{'}\right)k^2\alpha\beta + \mu_0^{\,2}k^2 L_2} \tag{29}$$

where

$$L_2 = \left(\frac{J_1^{''}(kr)}{J_1(kr)} + \frac{J_1^{'}(kr)}{krJ_1(kr)} + \frac{c^2}{\beta_{02}^2} - \frac{1}{k^2r^2} \right) \quad \text{and} \quad \beta_{02} = \sqrt{\frac{\mu_0}{\rho_2}}$$

Equation (29) is the dispersion equation for the propagation of torsional surface wave in isotropic layer sandwiched between initially stressed isotropic half-spaces.

Case III:

When homogeneous isotropic layer is lying over homogeneous isotropic half-space, i.e. $\mu_1 \to 0$, $\mu_2 = \mu$, $\mu_1^{'} = 0 = \mu_2^{'}$, $\mu_L = \mu_T = \mu_0$ and $P = 0$, then Eq. (35) becomes

$$\tan\left(kH\sqrt{\frac{c^2}{\beta_{02}^2} - 1} \right) = \frac{\mu\sqrt{1-\frac{c^2}{\beta_3^2}}}{\mu_0\sqrt{\frac{c^2}{\beta_{02}^2} - 1}} \tag{30}$$

This is the classical Love wave equation.

Numerical results and discussion

For the purpose of numerical computation of dimensionless phase velocity $\left(\frac{c}{\beta_2}\right)$ of torsional surface wave propagating in a viscoelastic medium over a composite layer lying over an initially stressed viscoelastic half-space, we have considered the following data:

(i) For the composite layer (Markham, 1991)

$\mu_L = 7.07 \times 10^9\ \mathrm{N/m^2}$, $\mu_T = 3.5 \times 10^9\ \mathrm{N/m^2}$, $\rho = 1600\, Kg/m^3$.

(ii) For the viscoelastic half- space (Gubbins, 1990)

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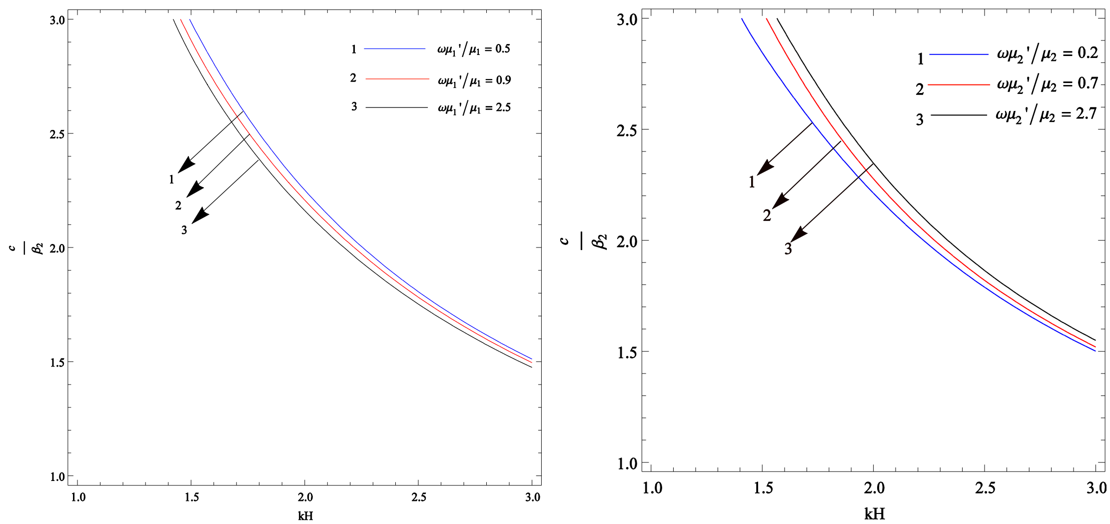
$\mu_1 = 6.77 \times 10^{10}\ \mathrm{N/m^2}, \rho_1 = 3323\, Kg/m^3$

$\mu_2 = 7.1 \times 10^{10}\ \mathrm{N/m^2}, \rho_2 = 3321\, Kg/m^3$



(2a) **(2b)**

Fig. 2 Variation in dimensionless phase velocity against dimensionless wave number for different values of angular frequency

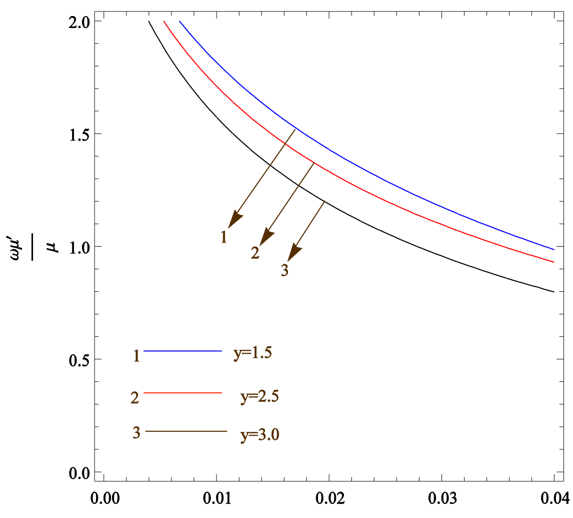


Fig. 3 Variation in angular frequency ($\omega\mu_1'/\mu_1$) against dimensionless wave number kH for different values of y, when $\omega\mu_2'/\mu_2 = 2.5$.

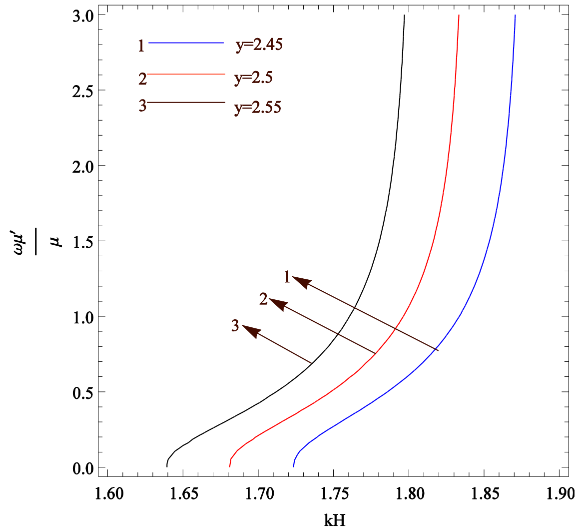


Fig. 4 Variation in angular frequency $\omega\mu_2{}'/\mu_2$ against dimensionless wave number kH for different values of y when $\omega\mu_1{}'/\mu_1 = 1.0$.

CONCLUSION

Torsional surface waves in self-reinforced layer sandwiched between pre-stressed viscoelastic half-spaces has been studied. Dispersion relation is obtained in closed form. Obtained dispersion relation has been match with classical Love wave equation for isotropic layer over an isotropic half-space. Graphs have been plotted between phase velocity and wave number and it is observed that the phase velocity decreases as increases the angular frequency of Torsional waves.

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